SCHUR-WEYL DUALITY FOR $U(n)$

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ABSTRACT. These are notes from a lecture given in MATH 26700, Introduction to Representation Theory of Finite Groups, at the University of Chicago in December 2009. Most of the material comes from chapter 8 of [Ful97].

1. Introduction

Let's recall the theory of weights for representations of $U(n)$. If we restrict a representation $R : U(n) \to GL(V)$ to the maximal torus $T \subset U(n)$, which consists of all diagonal matrices in $U(n)$, V decomposes as the sum of one-dimensional representations of T. In other words, we can write

$$V = \bigoplus_{\lambda \in \mathbb{Z}^n} V(\lambda),$$
$$R|_T(e^{i\theta}) = \bigoplus_{\lambda \in \mathbb{Z}^n} P_\lambda e^{i\lambda\cdot\theta},$$

where P_λ is projection onto $V(\lambda)$, and $e^{i\theta}$ is shorthand for

$$\begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} \in U(n).$$

The weights of V are those $\lambda \in \mathbb{Z}^n$ for which $d_\lambda = \dim V(\lambda) > 0$. We say that the weight λ has **multiplicity** d_λ in V.

The big result, which follows from a computation with Weyl's integral formula, is that V is irreducible if and only if there is a unique highest weight λ, and this weight has multiplicity 1. Furthermore, two irreducible representations are equivalent if and only if they have the same highest weight. Another computation shows that every dominant weight (i.e., a weight in which $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$) occurs as a highest weight for an irreducible representation of $U(n)$.

The next task is to construct irreducible representations associated with each highest weight. First, we will look at ways to construct new representations from old ones. Then, we will see that all irreducible representations of $U(n)$ can be built up from just one representation, the defining representation, where $V = \mathbb{C}^n$ and $U(n)$ acts by its standard embedding into $GL_n(\mathbb{C})$.

2. Some Linear Algebra

Let V and W be vector spaces. Recall that the **tensor product** $V \otimes W$ of V and W is a vector space equipped with a bilinear map $V \times W \to V \otimes W$. This vector space satisfies the following universal property: any bilinear map $V \times W \to U$ factors uniquely through a linear map $V \otimes W \to U$. In other words, $V \otimes W$ is the vector space with the property that a linear map out of $V \otimes W$ is the same as a bilinear map out of $V \times W$.

Explicitly, $V \otimes W$ can be constructed by taking the vector space $F(V \times W)$ freely generated by the symbols $v \otimes w$ for every $v \in V$, $w \in W$, and quotienting $F(V)$ by the following relations.

(1) $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$;
(2) $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$;
(3) $(av) \otimes w = a(v \otimes w) = v \otimes (aw)$ for $a \in \mathbb{C}$.

Exercise 2.1. (1) *Verify that $F(V)/\sim$ satisfies the universal property for the tensor product.*
(2) *Show that if $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ are bases for V and W, respectively, then $\{e_i \otimes f_j\}_{(i,j) \in I \times J}$ is a basis for $V \otimes W$. Thus, if V and W are finite dimensional,* $\dim(V \otimes W) = \dim V \times \dim W$.

Similarly, given m vector spaces $V_1, V_2, \ldots, V_m$, we can define the m-fold tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_m$ to be the vector space admitting a universal m-linear map from $V_1 \times \cdots \times V_m$. For any vector space V, we write $V^{\otimes m} = \underbrace{V \otimes \cdots \otimes V}_{m \text{ times}}$. This vector space is called the m**th tensor power** of V.

The explicit construction of m-fold tensor products and tensor powers is similar to the construction of $V \otimes W$. The basic building blocks are the elements of the form $v_1 \otimes \cdots \otimes v_m$, which are often called **simple tensors**. Note that in general, an element of $V_1 \otimes \cdots \otimes V_m$ will be a *finite linear combination* of simple tensors.

Exercise 2.2. *Find generators and relations and bases for m-fold tensor products and tensor powers.*

The mth tensor power $V^{\otimes m}$ carries a natural representation of the symmetric group S_m. On the level of $F(V^{\times m})$, $\sigma \in S_m$ sends $v_1 \otimes \cdots \otimes v_m$ to $v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}$.

Exercise 2.3. *Verify that this action of S_m descends to an representation on $V^{\otimes m}$.*

Actually, it is often more convenient to think of the action of S_m on $V^{\otimes m}$ as a **right representation**, i.e., to view $V^{\otimes m}$ as a right $\mathbb{C}S_m$-module. We define

$$(v_1 \otimes \cdots \otimes v_m) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)}.$$

The switch from a left action to a right action allows us to replace σ^{-1} by σ.

Two other common constructions are the symmetric and alternating powers of a vector space V. The m**th symmetric power** $\operatorname{Sym}^m V$ of V is defined to be the quotient of $V^{\otimes m}$ by the action of all $\sigma \in S_m$. In other words, we identify $v_1 \otimes \cdots \otimes v_m$ and $w_1 \otimes \cdots \otimes w_m$ if there exists $\sigma \in S_m$ such that $w_i = v_{\sigma(i)}$ for all i. We write a simple tensor in Sym^m as $v_1 v_2 \cdots v_m$.

Exercise 2.4. *Show that choosing a basis $\{e_i\}_{i \in I}$ for V gives an identification of* $\operatorname{Sym}^m(V)$ *with homogeneous polynomials of degree m in the variables e_i. If* $\dim V = n$, *what is* $\dim \operatorname{Sym}^m(V)$*?*

Similarly, the m**th alternating power** $\Lambda^m V$ of V is defined to be the quotient of $V^{\otimes m}$ by the action of S_n in which $\sigma \in S_n$ sends $v_1 \otimes \cdots \otimes v_m$ to $(-1)^{\operatorname{sgn} \sigma} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)}$. We write a simple tensor of $\Lambda^m V$ as $v_1 \wedge \cdots \wedge v_m$.

Exercise 2.5. *If* $\dim V = n$, *what is* $\dim \Lambda^m(V)$*?*

We now generalize the constructions of tensor, symmetric, and alternating powers. These generalizations are called **Schur functors**.

Let M be a representation of S_m (i.e., a left $\mathbb{C}S_m$-module). We can form the tensor product $\mathbb{S}_M(V) = V^{\otimes m} \otimes_{\mathbb{C}S_m} M$, which is the quotient of $V^{\otimes m} \otimes M$ by the subspace generated by

$$(x \cdot \sigma) \otimes y - x \otimes (\sigma \cdot y), \quad x \in V^{\otimes m}, \sigma \in S_m, y \in M.$$

Exercise 2.6.
(1) *If $M = \mathbb{C}S_m$ is the regular representation, then $\mathbb{S}_M(V) = V^{\otimes m}$.*
(2) *If $M = S^{(m)}$ is the trivial representation, then $\mathbb{S}_M(V) = \operatorname{Sym}^m V$.*
(3) *If $M = S^{(1,1,\ldots,1)}$ is the sign representation, then $\mathbb{S}_M(V) = \Lambda^m V$.*

For $M = S^\lambda$ a Specht module, we denote the corresponding Schur functor by $\mathbb{S}^\lambda$.

3. Decomposing Tensor Powers

We now turn to the problem of constructing new representations from old ones using tensor powers.

Let V be a representation of a group G. Then $V^{\otimes m}$ also has the structure of a representation of G. On simple tensors, $g \in G$ acts by

$$g \cdot (v_1 \otimes \cdots \otimes v_m) = (g \cdot v_1) \otimes (g \cdot v_2) \otimes \cdots \otimes (g \cdot v_m).$$

Furthermore, any Schur functor applied to V will have the structure of a representation of G; namely, we define

$$g \cdot (x \otimes y) = (g \cdot x) \otimes y$$

for $x \otimes y \in V^{\otimes m} \otimes_{\mathbb{C}S_m} M$.

From Exercise 2.6, we have the rather tautologous identity

$$V^{\otimes m} = \mathbb{S}_{\mathbb{C}S_m}(V) = V^{\otimes m} \otimes_{\mathbb{C}S_m} \mathbb{C}S_m.$$

Here we think of the $\mathbb{C}S_m$ on the right as a $\mathbb{C}S_m$-bimodule, that is, a module over $\mathbb{C}S_m$ by multiplication on both the left and the right. The resulting tensor product $V^{\otimes m}$ then carries an action of G on the left (as above) and an action of $\mathbb{C}S_m$ on the right. The key thing to note is the because we have G acting only on the left factor $V^{\otimes m}$ and $\mathbb{C}S_m$ acting only on the right factor $\mathbb{C}S_m$, the actions of G and $\mathbb{C}S_m$ commute with each other.

Now, recall that the Peter-Weyl theorem gives us a decomposition of the left (or right) regular representation of S_m into the direct sum of irreducibles, with each irreducible appearing as many times as its dimension. However, what we want to do here is to decompose $\mathbb{C}S_m$ as a $\mathbb{C}S_m$-bimodule. This will then give us a decomposition of $V^{\otimes m} = \mathbb{S}_{\mathbb{C}S_m}(V)$ as a right $\mathbb{C}S_m$-bimodule. A slightly stronger statement of the Peter-Weyl theorem says that as a $\mathbb{C}S_m$-bimodule (under the left and right regular actions), $\mathbb{C}S_m \cong \bigoplus_{L \in \widehat{S_m}} L \otimes L^*$. Here, L^* is the **dual representation** to L, which is a representation defined on the dual space to L. The left action of $\mathbb{C}S_m$ acts on the left factor L, and the right action acts on L^*.

We have classified the irreducible representations of S_m; they are S^λ for $\lambda \vdash m$. These representations are self-dual, so we can write

$$\mathbb{C}S_m = \bigoplus_{\lambda \vdash m} S^\lambda \otimes S^\lambda.$$

Plugging this in to the formula for $V^{\otimes m} = \mathbb{S}_{\mathbb{C}S_m}(V)$ gives us a decomposition

$$\begin{aligned} V^{\otimes m} &= V^{\otimes m} \otimes_{\mathbb{C}S_m} \left(\bigoplus_{\lambda \vdash m} S^\lambda \otimes S^\lambda \right) \\ &= \bigoplus_{\lambda \vdash m} V^{\otimes m} \otimes_{\mathbb{C}S_m} (S^\lambda \otimes S^\lambda) \\ &= \bigoplus_{\lambda \vdash m} (V^{\otimes m} \otimes_{\mathbb{C}S_m} S^\lambda) \otimes S^\lambda \\ &= \bigoplus_{\lambda \vdash m} \mathbb{S}^\lambda(V) \otimes S^\lambda. \end{aligned}$$

This is a decomposition as a right $\mathbb{C}S_m$-module, but since the action of $\mathbb{C}S_m$ commutes with that of G, it is in fact also a decomposition as a representation of G.

Exercise 3.1. *Show that* $\mathbb{S}^\lambda(V)$ *is nonzero if and only if* λ *has at most* $\dim V$ *parts.*

We conclude that the representation $V^{\otimes m}$ decomposes as the direct sum of the $\mathbb{S}^\lambda(V)$ for each $\lambda \vdash m$ with at most $\dim V$ parts, and each $\mathbb{S}^\lambda(V)$ appears in this decomposition $\dim S^\lambda$ times.

We've said about all that we can say for the general situation. This is because in general, the representations $\mathbb{S}^\lambda(V)$ do not need to be irreducible. However, when $G = U(n)$ and $V = \mathbb{C}^n$ as above, it turns out that they are. In fact, we have the following result.

Theorem 3.2. $\mathbb{S}^\lambda(\mathbb{C}^n)$ *is the irreducible representation of* $U(n)$ *with highest weight* λ.

We don't have time to prove this fact, which takes a bit of work. A suggested reference is chapter 8 of [Ful97].

The possible highest weights for an irreducible representation of $U(n)$ are the **dominant weights**, that is, the weights λ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. If all of the λ_i are nonnegative, then λ is a partition of $m = \sum_{i=1}^n \lambda_i$ with (at most) n parts, so $\mathbb{S}^\lambda(\mathbb{C}^n)$ is the desired irreducible representation.

To get all other representations, we make use of a one-dimensional representation $V_{\det}$ of $U(n)$ called the **determinant representation**. In this representation, an element $g \in U(n)$ acts as multiplication by $\det g$.

Exercise 3.3. *Show that the determinant representation is equivalent to* $\Lambda^n(\mathbb{C}^n) \cong \mathbb{S}^{(1,1,\ldots,1)}(\mathbb{C}^n)$.

The dual representation to the determinant representation is $V_{\det}^*$, in which g acts as multiplication by $(\det g)^{-1}$. We will denote this representation by $V_{\det}^{\otimes -1}$, and its kth tensor power by $V_{\det}^{\otimes -k}$.

Exercise 3.4.
(1) *Show that* $V_{\det}^{\otimes i} \otimes V_{\det}^{\otimes j} \cong V_{\det}^{\otimes i+j}$ *for any* $i, j \in \mathbb{Z}$.
(2) *Show that if* V *is an irreducible representation of highest weight* λ*, then* $V \otimes V_{\det}^{\otimes k}$ *is an irreducible representation of highest weight* $\lambda + k = (\lambda_1 + k, \lambda_2 + k, \ldots, \lambda_n + k)$.

From this exercise, we conclude that we can get any irreducible representation of $U(n)$ by tensoring some $\mathbb{S}^\lambda(\mathbb{C}^n)$ with some negative power of the determinant representation.

The sum total of all of these results is known as **Schur-Weyl duality**. We summarize these facts as follows.

Theorem 3.5 (Schur-Weyl)**.** *As a representation of $U(n) \times S_m$,*

$$(\mathbb{C}^n)^{\otimes m} \cong \bigoplus_{\substack{\lambda \vdash m, \\ \ell(\lambda) \leq n}} L_\lambda \otimes S^\lambda,$$

where $\ell(\lambda)$ is the number of parts of λ, and L_λ is the irreducible representation of $U(n)$ with highest weight λ.

Corollary 3.6. *Any irreducible representation of $SU(n)$ appears in $(\mathbb{C}^n)^{\otimes m}$ for $m \geq n$. Any irreducible representation of $U(n)$ can be obtained from one in $(\mathbb{C}^n)^{\otimes m}$ after tensoring with sufficiently large negative power of the determinant representation.*

While we constructed the Schur functors using the Specht modules, there are various direct constructions of the Schur functors. See chapter 8 of [Ful97] for details.

References

[Ful97] William Fulton, *Young tableaux*, London Mathematical Society Student Texts, no. 35, Cambridge University Press, 1997.

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