REPRESENTATION THEORY, SYMMETRY, AND QUANTUM MECHANICS

EVAN JENKINS

ABSTRACT. While the equations governing physical systems often seem forbiddingly complicated, the symmetry that underlies the physical world can provide a beautiful insight into their solutions. In this talk, we introduce some basic notions of representation theory and use them to show how the "quantum" behavior of atomic electrons arises from symmetry. These notes are from a talk given on August 15, 2008 to mathematics undergraduates at the University of Chicago VIGRE REU.

1. INTRODUCTION

In the early 20th century, it had become apparent to physicists that many phenomena, from the orbiting of electrons in atoms to the emission and absorption of light waves, did not occur on a continuous spectrum, as classical theories would predict. Einstein's 1905 discovery of the photoelectric effect showed that light is emitted and absorbed in discrete packets called *quanta*, the energy of which is proportional to the frequency of the light. Bohr's 1913 model of the atom had electrons orbiting at a discrete set of distances from the nucleus. Early attempts to explain these phenomena consisted of taking the classical equations of physics and applying artificial constraints to make their solutions exhibit these observed quantum behaviors. Such attempts met with little success. It was not until Heisenberg and Schödinger rewrote the foundations of physics that the true nature of quantization was revealed: it arises naturally out of the symmetry of the equations governing the physics.

2. LIE GROUPS AND REPRESENTATIONS

The rules of symmetry are encoded in the theory of group representations. The archetypical symmetry group is the **general linear group** of a real or complex vector space V, denoted $GL(V)$. This group consists of all invertible linear transformations on V. A **representation** of a group G is a (smooth) group homomorphism from G to $GL(V)$. In other words, a representation realizes the elements of G as transformations of V in such a way that multiplication of elements of G corresponds to the composition of transformations. We will mainly be concerned with representations of **matrix Lie groups**, which are that can be realized as (topologically closed) subgroups of $GL(\mathbb{R}^n)$ for some n. We will write $GL_n(\mathbb{R})$ for $GL(\mathbb{R}^n)$.

An example that will be important for us is the **special orthogonal group** $SO(3)$. This is the subgroup of $GL_3(\mathbb{R})$ consisting of matrices that preserve lengths, angles, and orientation in $\mathbb{R}^3$. More precisely,

$$SO(3) = \{A \in GL_3(\mathbb{R}) \mid A^T A = I, \det A = 1\}. \tag{2.1}$$

As a subgroup of $GL_3(\mathbb{R})$, $SO(3)$ has a natural representation on $\mathbb{R}^3$ given by the inclusion map

$$\rho : SO(3) \hookrightarrow GL_3(\mathbb{R}). \tag{2.2}$$

In this representation, an element in $A \in SO(3)$ acts on a vector $\mathbf{r} \in \mathbb{R}^3$ by the usual matrix multiplication:

$$\rho(A)v = Av.$$

For a slightly less obvious example of a representation, consider the representation

$$\det : GL_n(\mathbb{R}) \to GL(\mathbb{R}) \cong \mathbb{R}^*$$

taking a matrix A to its determinant.

We often refer to a representation $\rho : G \to GL(V)$ by the space V alone and leave the map ρ implicit. Where there is no ambiguity, we write $g \cdot v$ for $\rho(g)v$.

Given two representations

$$\rho_1 : G \to GL(V_1),$$
$$\rho_2 : G \to GL(V_2)$$

of a group G, we can form the **direct sum** $V \oplus W$ of the two representations,

$$(\rho_1 \oplus \rho_2) : G \to GL(V_1 \oplus V_2)$$

via the natural inclusion $GL(V) \times GL(W) \to GL(V \oplus W)$. Note that if we pick bases of V and W and take their union as a basis of $V \oplus W$, $(\rho_1 \oplus \rho_2)(g)$ will have a block-diagonal matrix of the form

$$\begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}.$$

A **subrepresentation** of $\rho : G \to GL(V)$ is a subspace $W \subset V$ such that $\forall g \in G, w \in W, \rho(g)w \in W$. For instance, V_1 and V_2 are subrepresentations of the direct sum $V_1 \oplus V_2$ defined above. A representation containing no subrepresentations other than 0 and itself is called **irreducible**. It is easy to see that the representation det of $GL_n(\mathbb{R})$ and that of $SO(3)$ given above are irreducible. In sufficiently nice situations, a representation will decompose into the direct sum of irreducible representations. Thus, the problem of classifying representations reduces to that of classifying irreducible representations.

There is another viewpoint for representation theory that will prove useful for our study of quantum mechanics. The **Lie algebra** $\mathfrak{g}$ of a matrix Lie group G is the tangent space to G at the identity. $GL_n(\mathbb{R})$ is an open subset of the space of $n \times n$ matrices, $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, so its tangent space can be identified with $M_n(\mathbb{R})$. We will denote this tangent space by $\mathfrak{gl}_n\mathbb{R}$. The Lie algebra $\mathfrak{g}$ of G will be a subspace of $\mathfrak{gl}_n\mathbb{R}$. The Lie algebra carries an operation called the **Lie bracket**, defined by

$$[X, Y] = XY - YX, \quad X, Y \in \mathfrak{g}. \tag{2.3}$$

The Lie algebra of G will be a **Lie subalgebra** of $\mathfrak{gl}_n\mathbb{R}$, i.e., a subspace that is closed under the Lie bracket.

The intuition behind the Lie algebra is that its elements correspond to "infinitesimal transformations," to appropriate the physics lingo. If we think of elements of $GL_n(\mathbb{R})$ as transformations of $\mathbb{R}^n$, an element $X \in \mathfrak{g}$ should generate a one-parameter family of transformations whose derivative (in the appropriate sense) at

0 is X. This one parameter family is generated by **exponentiation**:

$$\phi_s(X) = e^{sX} = \sum_{i=0}^{\infty} \frac{s^i X^i}{i!}.$$

One can then check that this family of transformations indeed lies in G, and that its derivative at $s = 0$ is X.

Given a group representation $\rho : G \to GL(V)$, we can differentiate it at the identity to get a **Lie algebra representation** $\rho_* : \mathfrak{g} \to \mathfrak{gl}(V)$. Explicitly,

$$\rho_*(X)v = \frac{d}{ds} \left(\rho(e^{sX})v \right) \Big|_{s=0}. \tag{2.4}$$

We will be looking at representations of $\mathfrak{so}_3$, the Lie algebra of $SO(3)$, which consists of 3×3 matrices X with trace 0 that satisfy $X^T = -X$. (It is instructive to investigate how this definition of $\mathfrak{so}(3)$ relates to the definition (2.1) of $SO(3)$.) It turns out that the finite-dimensional irreducible representations of $\mathfrak{so}(3)$ on complex vector spaces are easy to classify.[1] For every nonnegative half-integer ℓ, there is exactly one (up to isomorphism) irreducible representation $\rho_* : \mathfrak{so}(3) \to \mathfrak{gl}(V_\ell)$ of dimension $2\ell + 1$. Furthermore, if we let

$$L_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(3), \tag{2.5}$$

then $\rho_*(L_z)$ is diagonalizable, with $2\ell + 1$ distinct eigenvalues

$$\lambda_1 = \ell,$$
$$\lambda_2 = \ell - 1,$$
$$\lambda_3 = \ell - 2,$$
$$\vdots$$
$$\lambda_{2\ell} = -\ell + 1,$$
$$\lambda_{2\ell+1} = -\ell. \tag{2.6}$$

The one-parameter family in $SO(3)$ generated by L_z acts as rotation about the z-axis in the natural representation (2.2) on $\mathbb{R}^3$.

In addition to classifying representations of a Lie algebra, we can also determine the **branching rules**. Namely, given a Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$, an irreducible representation V of $\mathfrak{g}$ can also be thought of as a representation of $\mathfrak{g}'$, but it may no longer be irreducible. So we can ask which irreducible representations of $\mathfrak{g}'$ occur in a given irreducible representation of $\mathfrak{g}$. More generally, we may ask this question for a chain $\mathfrak{g} \supset \mathfrak{g}' \supset \mathfrak{g}'' \supset \cdots$ of Lie subalgebras. This is the branching problem for $\mathfrak{g}$.

For $\mathfrak{so}(3)$, any nonzero proper Lie subalgebra is isomorphic to the one-dimensional subalgebra $\mathfrak{h}$ consisting of all scalar multiples of L_z. Thus our characterization (2.6) of the eigenvalues of L_z on V_m solves the branching problem for $\mathfrak{so}(3)$: the

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[1] We are being a little sloppy here, as the representations we will consider later come from representations of $SO(3)$, and not every finite-dimensional representation of $\mathfrak{so}(3)$ comes from a representation of $SO(3)$. This difficulty occurs because $SO(3)$ is not simply connected. However, in quantum mechanics, what we really want to consider are **projective representations**, which are maps $\rho : G \to PGL(V)$. Every finite-dimensional representation of $\mathfrak{so}(3)$ will come from a projective representation of $SO(3)$, so we do not need to make a distinction.

representation V_ℓ of $\mathfrak{so}(3)$ contains the (necessarily one-dimensional) irreducible representations $W_\ell, W_{\ell-1}, \ldots, W_{-\ell}$ of $\mathfrak{h}$, where

$$(kL_z) \cdot w = (km)w, \quad k \in \mathbb{R}, m \in W_m. \tag{2.7}$$

One important basic tool of representation theory is the following.

Lemma 2.8 (Schur). *If $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is an irreducible representation on a complex vector space V, and $T : V \to V$ is a linear transformation such that $\forall X \in \mathfrak{g}, v \in V$,*

$$T(X \cdot v) = X \cdot (Tv),$$

then $\exists k \in \mathbb{C}$ such that $Tv = kv$ for all $v \in V$.

Proof. Pick an eigenvector v_0 of T of eigenvalue k. Let

$$W = \{v \in V \mid Tv = kv\}.$$

If $w \in W$, then

$$\begin{aligned} T(X \cdot w) &= X \cdot (Tw) \\ &= X \cdot (kw) \\ &= k(X \cdot w), \end{aligned}$$

and hence $X \cdot w \in W$. This proves that W is a subrepresentation of V. Since W is nonempty, $W = V$ by irreducibility, and thus $Tv = kv$ for all $v \in V$ as desired. $\square$

3. Quantum Mechanics

Classical mechanics is governed by Newton's laws. One equivalent formulation of these laws is that of Hamiltonian mechanics. This formulation has two types of variables: **position variables** and **momentum variables**. We will be considering the case of a single particle moving, in which case there are three position variables, x, y, and z, which denote the coordinates of the particle's location in $\mathbb{R}^3$, and three momentum variables p_x, p_y, and p_z, which denote the three components of the **linear momentum**, where momentum in a direction is just the mass m of the particle multiplied by the particle's velocity in that direction. We write $\mathbf{r}$ for the position vector

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

and $\mathbf{p}$ for the momentum vector

$$\mathbf{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}.$$

Hamilton's equations relate the time derivatives of these variables to a function of them called the **Hamiltonian**, which measures the total energy of the system. A mechanical system has two types of energy, **kinetic energy** and **potential energy**. Kinetic energy is the energy imbued in the particle's motion, and it is given by

$$K(\mathbf{p}) = \frac{\mathbf{p} \cdot \mathbf{p}}{2m}. \tag{3.1}$$

Potential energy, roughly speaking, is a measure of the energy stored in a system. For instance, if our system is a ball sitting on the ground, then lifting the ball off the ground will add gravitational potential energy, which will change into kinetic

energy if the ball is then dropped. The potential energy of such a system will be a function of the height of the ball off the ground. In general, potential energy is given by some function $U(\mathbf{r})$. Thus, we can write the Hamiltonian as

$$H(\mathbf{r}, \mathbf{p}) = K(\mathbf{p}) + U(\mathbf{r}). \qquad (3.2)$$

In quantum mechanics, the notion of a particle having a fixed position and momentum at a given time is swept off the table. Instead, its state will be described probabilistically by a **wave function** $\psi \in L^2(\mathbb{R}^3)$, the (infinite-dimensional) Hilbert space of square-integrable complex-valued functions in $\mathbb{R}^3$, which we will denote by $\mathcal{H}$. $\mathcal{H}$ has an inner-product on it given by

$$\langle \psi_1, \psi_2 \rangle = \int_{\mathbb{R}^3} \psi_1(x, y, z) \overline{\psi_2(x, y, z)} \, dx \, dy \, dz. \qquad (3.3)$$

The basic interpretation of the wave function is that its absolute square $|\psi|^2$, when properly normalized, is a probability distribution for the location of the particle.

Every measurable quantity of a quantum mechanical system is associated with an **observable**, a self-adjoint operator T on $\mathcal{H}$ (or more precisely, on some dense subspace of $\mathcal{H}$). Self-adjoint operators are diagonalizable, i.e., there exists an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors of T. These eigenvectors are called **eigenstates** for T. Perhaps the most bizarre feature of quantum mechanics is that if we take some measurement of a system, such as determining its position or momentum, we always find that it is in an eigenstate for the corresponding observable. The eigenvalue of this eigenstate is the value that we measure. Until we measure the system, ψ will be in a **superposition** of eigenstates, and the probability that our measurement will find a system with initial state ψ in an eigenstate ψ_λ is proportional to $|\langle \psi, \psi_\lambda \rangle|^2$.

To formulate quantum mechanics, we must give observables corresponding to the position and momentum variables of classical mechanics. The **position operators** x, y, and z are simply multiplication by x, y, and z, respectively, while p_x is replaced by the **momentum operator** $-i\frac{\partial}{\partial x}$, and similarly for p_y and p_z.[2] While these operators are not defined on all of $\mathcal{H}$, they are defined on a dense subspace.[3]

The quantum analog of Hamilton's equations is **Schrödinger's equation**, which governs the time evolution of the wave function ψ, where we now consider ψ as a function also of time such that at any time t, $\psi(-, t) \in \mathcal{H}$. For a general Hamiltonian H (where we treat H now as an operator on $\mathcal{H}$ by replacing the position and momentum variables by their corresponding operators), it is given by

$$\frac{\partial \psi}{\partial t} = -iH\psi. \qquad (3.4)$$

Thus, to determine the behavior of a quantum system, one must solve Schrödinger's equation for the appropriate Hamiltonian H. If $\psi_0 = \psi(x, y, z, 0)$ is an eigenstate for H (an **energy eigenstate**) of eigenvalue λ, then the time evolution of ψ is clear:

$$\psi(\mathbf{r}, t) = \psi_0(\mathbf{r}) e^{-i\lambda t}. \qquad (3.5)$$

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[2]Astute physics students should note that we are working in units with $\hbar = 1$.

[3]In order to diagonalize such operators, we must extend $\mathcal{H}$ to include certain distributions. Physicists call such an extension a **rigged Hilbert space**. We need not be concerned with the technical details.

We note that (3.5) is a statement of **conservation of energy**: if a system is in an energy eigenstate at any one time, it remains in an energy eigenstate at all times.

The well-behaved evolution of energy eigenstates reduces the problem of solving Schrödinger's equation to that of diagonalizing the operator H. Unfortunately, this can be difficult or impossible to do analytically for complicated Hamiltonians. And in simpler cases, we are often confronted with **degeneracy**: multiple independent energy eigenstates with the same eigenvalue. But degeneracy often indicates the presence of symmetry, and we can make use of this symmetry to study the solutions to Schrödinger's equation.

If $\rho : G \to GL(\mathbb{R}^3)$ is a representation of a group G on $\mathbb{R}^3$, then we get an associated representation $\tilde{\rho}$ of G on $\mathcal{H}$ by setting

$$\tilde{\rho}\psi(\mathbf{r}, t) = \psi(\rho(g^{-1})\mathbf{r}, t). \tag{3.6}$$

Note that our original representation ρ was on the real vector space $\mathbb{R}^3$, while $\tilde{\rho}$ is a representation on the complex vector space $\mathcal{H}$. We say that such a representation is a **symmetry** of the Hamiltonian H if for every $g \in G$, $\psi \in \mathcal{H}$, we have

$$H(g \cdot \psi) = g \cdot (H\psi) \tag{3.7}$$

Denote by $\mathcal{A}_H$ the subspace of $\mathcal{H}$ consisting of solutions to Schrödinger's equation for the Hamiltonian H. Then if $\psi \in \mathcal{A}_H$,

$$
\begin{aligned}
-iH(g \cdot \psi) &\overset{(3.7)}{=} -i(g \cdot (H\psi)) \\
&\overset{(3.4)}{=} g \cdot \left(\frac{\partial}{\partial t}\psi\right) \\
&= \frac{\partial}{\partial t}(g \cdot \psi),
\end{aligned}
\tag{3.8}
$$

and thus $g \cdot \psi \in \mathcal{A}_H$. We have thus proved that if a representation ρ is a symmetry of H, then its associated representation $\tilde{\rho}$ on $\mathcal{H}$ has the subspace of solutions $\mathcal{A}_H$ as a subrepresentation. We also obtain a corresponding Lie algebra representation $\tilde{\rho}_* : \mathfrak{g} \to \mathfrak{gl}(\mathcal{A}_H)$. But we can say even more, as it is clear from (3.7) that if $\psi \in \mathcal{A}_H$ is an energy eigenstate with eigenvalue λ, then so is $g \cdot \psi$, so we can even restrict $\tilde{\rho}$ and $\tilde{\rho}_*$ to representations on $\mathcal{A}_{H,\lambda}$, the space of energy eigenstates with eigenvalue λ.

We now turn to a specific case: the electron of a hydrogen atom. A hydrogen atom consists of a positively charged nucleus, which we will assume to be fixed at the origin in $\mathbb{R}^3$, surrounded by a negatively charged electron. Updating the classical expression (3.1) for the kinetic energy to a quantum mechanical operator by substituting $-i\nabla$ for p gives

$$K = -\frac{1}{2m}\nabla^2. \tag{3.9}$$

The theory of electromagnetism tells us that the potential energy will be proportional to the inverse of the distance to the nucleus and negative, so we have

$$U = -\frac{k}{r}, \quad k \in \mathbb{R}^+, r = \sqrt{x^2 + y^2 + z^2}. \tag{3.10}$$

As in the classical case (3.2), combining (3.9) and (3.10) gives

$$H = K + U = -\frac{1}{2m}\nabla^2 - \frac{k}{r}. \tag{3.11}$$

It turns out that the spaces $\mathcal{A}_{H,\lambda_n}$, n a positive integer, of energy eigenstates for this system are finite dimensional of dimension n^2. We will write these spaces as $\mathcal{A}_n$ from now on.

Now we look at some symmetries of this Hamiltonian. Throughout, we will take $\psi \in \mathcal{A}_n$.

The simplest example of a symmetry of H is its symmetry under time translation. Namely, the group $G = \mathbb{R}$, whose Lie algebra is $\mathfrak{g} = \mathbb{R}$ (and whose "exponential map" is the identity map), acts on $\mathcal{H}$ by

$$\tilde{\rho}(g)\psi(\mathbf{r}, t) = \psi(\mathbf{r}, t - g), \quad g \in \mathbb{R}.$$

Note that while this representation is not a symmetry of the form (3.6), it is clear that it still satisfies the invariance property (3.8). We now look at the action of an element $X \in \mathbb{R}$ of the Lie algebra:

$$\tilde{\rho}_*(X)\psi \overset{(2.4)}{=} \left. \frac{d}{ds}\psi(\mathbf{r}, t - sX)\right|_{s=0}$$
$$= -X\frac{d\psi}{dt}(\mathbf{r}, t)$$
$$\overset{(3.4)}{=} iXH\psi(\mathbf{r}, t).$$

We see by the evolution equation (3.5) for energy eigenstates that eigenvectors of $\tilde{\rho}_*(X)$ in $\mathcal{A}_n$ will remain eigenvectors with the same eigenvalues at all times. Of course, in this case, $\tilde{\rho}_*(X)$ is just an imaginary scalar multiple of the Hamiltonian, so we have reproved conversation of energy. But we derived it by using a symmetry, the time-invariance of the Hamiltonian. This is a special case of **Noether's theorem**, which sets up a correspondence between symmetries of a physical system and conserved quantities.

We now look at a much less trivial example. We take $G = SO(3)$, $\mathfrak{g} = \mathfrak{so}(3)$, and ρ the natural representation (2.2) on $\mathbb{R}^3$. It is clear that ρ is a symmetry of the Hamiltonian (3.11), as neither the operator ∇^2 nor the distance from the origin r are changed by a rotation about the origin. We look at the action of the element L_z (2.5) of $\mathfrak{so}(3)$ on an element $\psi \in \mathcal{A}_n$.

$$(L_z \cdot \psi)(\mathbf{r}, t) \overset{(2.4)}{=} \left. \frac{d}{ds}\psi(e^{-sL_z}\mathbf{r}, t)\right|_{s=0}$$
$$= \sum_{w \in \{x,y,z\}} \frac{\partial\psi}{\partial w}\left(\left.\frac{d}{ds}(e^{-sL_z}\mathbf{r})\right|_{s=0}\right)_w$$
$$= \sum_{w \in \{x,y,z\}} (-L_z\mathbf{r})_w \frac{\partial\psi}{\partial w}$$
$$= -y\frac{\partial\psi}{\partial x} + x\frac{\partial\psi}{\partial y}.$$

Physicists will recognize the operator

$$i\tilde{\rho}_*(L_z) = iy\frac{\partial}{\partial x} - ix\frac{\partial}{\partial y} \tag{3.12}$$

as the z component of **angular momentum**, which, roughly speaking, is a measure of how much rotational motion a system has about the origin.[4] It is given classically by $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. The same computation with

$$L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad L_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

in place of L_z yields the x and y components of angular momentum, respectively. By our discussion of Noether's theorem above, we have derived **conservation of angular momentum**.

Finally, we are ready to demonstrate the "quantum" aspect of quantum mechanics. We have seen that the finite-dimensional space $\mathcal{A}_n$ carries a representation of $\mathfrak{g} = \mathfrak{so}(3)$, so we may decompose it into a direct sum of finite-dimensional irreducible representations. To every Lie algebra are associated certain algebraic combinations of elements called **Casimir operators** that have the special property that they commute with every element of the Lie algebra. Any such element, by Lemma 2.8, must act as multiplication by a scalar under any irreducible representation. To see what exactly we mean by this, we look at an important example. One Casimir operator for $\mathfrak{so}(3)$ is

$$L^2 = L_x^2 + L_y^2 + L_z^2.$$

If $\phi : \mathfrak{so}(3) \to \mathfrak{gl}(V)$ is any irreducible representation, then to say that L^2 is a scalar multiple of the identity under ϕ is to say that there is some $\lambda_{L^2} \in \mathbb{C}$ such that, for all $v \in V$,

$$\phi(L_x)\phi(L_x)v + \phi(L_y)\phi(L_y)v + \phi(L_z)\phi(L_z)v = \lambda_{L^2}v.$$

It turns out that the value of λ_{L^2} for this Casimir operator is enough to determine an irreducible representation of $\mathfrak{so}(3)$ uniquely. In particular, L^2 acts on V_ℓ as multiplication by $\ell(\ell+1)$. Recall that when we observe the square magnitude of the angular momentum of ψ, we will find it in an eigenstate for L^2, so an observed wave function ψ will lie in only one of the irreducible representations and thus have a well-defined value of ℓ. The absolute angular momentum of ψ will be the square root of the L^2-eigenvalue, $\sqrt{\ell(\ell+1)}$, so we have shown that angular momentum is **quantized**, i.e., it can only take on a discrete set of values.

Yet we can say even more by applying the branching rules. Inside $\mathfrak{so}(3)$ sits a one-dimensional Lie algebra $\mathfrak{h}$ generated by L_z. The element L_z itself is a Casimir element for this algebra, whose corresponding eigenvalue is the **magnetic quantum number** m_ℓ. Thus, the z component of angular momentum is quantized as well. We see from the formulation (2.7) of the $\mathfrak{h}$-subrepresentations W_m in V_ℓ that m_ℓ can range from $-\ell$ to ℓ in integer steps.

4. Conclusion

By studying the $SO(3)$ symmetry of the hydrogen atom, we have shown the quantization of angular momentum and its z component and demonstrated the relationship between the ℓ and m_ℓ quantum numbers. But any physics student would certainly object that we have not shown the quantization of energy or the quantization of spin, which is an internal angular momentum possessed by most quantum particles. But with a little more work, the same basic ideas will give us

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[4]Physicists throw in a factor of i to switch from the anti-self-adjoint operator $\rho_*(L_z)$ to the self-adjoint operator $i\rho_*(L_z)$, but this distinction scarcely matters from our point of view.

all of these quantum numbers. What we have neglected is that particles exist not just in three-dimensional space but in a relativistic **spacetime**, and we can gain more information by looking a not only separate symmetries of space and time, but symmetries of spacetime.

We have only demonstrated the very tip of the iceberg when it comes to the connections between representation theory and physics. Our construction of the hydrogen atom was rather *ad hoc*, as we assumed the presence of a nucleus. If we begin with an empty universe, then particles can be viewed as representations of the **Poincaré group**, which is the symmetry group of the universe. In addition to these global symmetries, the universe also has local symmetries that describe the forces of nature. The study of these local symmetries is called **gauge theory**. This framework is the foundation for the **standard model of particle physics**. The standard model, while extremely successful at describing the workings of the atomic and subatomic world, unfortunately does not incorporate the theory of gravity. Governed by Einstein's general relativity, gravity bends spacetime and thus robs us of the power of Poincaré group symmetries. One of the most important problems in theoretical physics today is to reconcile particle physics with general relativity, a task that appears to require some very deep mathematics.

A popular, reasonably accessible introduction to the representation theory of Lie algebras is the book of Humphreys, [3]. The standard introductory textbook for quantum mechanics is Griffiths, [2], but as with most physics textbooks, its lack of mathematical rigor can be offputting. A fairly rigorous treatment of quantum field theory is given in Ticciati, [5], but one should first have a firm understanding of representation theory, functional analysis, differential geometry, and the basics of quantum mechanics before attempting such a course of study. For an excellent and elementary introduction of the mathematics of gauge theory as well as its relevance to both physics and mathematics, see Baez and Muniain, [1].

References

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