ÉTALE COHOMOLOGY SEMINAR LECTURE 1

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1. Introduction

The theory of étale cohomology springs from a simple question: is it possible to do algebraic topology on algebraic varieties (or, more generally, schemes)? It is often taken for granted today that a positive resolution to this question was once very much in doubt. The Zariski topology is, from the point of view of a topologist, simply atrocious. The basic building blocks of algebraic topology–simplices, closed loops, universal covers–are simply not available in the general setting of algebraic varieties. Consider the "nice" case of varieties over $\mathbb{C}$: spheres of odd dimension do not even exist as objects! The Zariski topology is completely oblivious to standard topological information: the higher cohomology of a constant sheaf on an irreducible scheme is zero!

The one saving grace of the Zariski topology, at least over $\mathbb{C}$, is Serre's GAGA theorem (and other similar results), which states that, for a coherent sheaf $\mathcal{F}$ on a projective variety X over $\mathbb{C}$, there is an isomorphism

$$H^i(X, \mathcal{F}) \cong H^i(X^{\mathrm{an}}, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\mathrm{an}}).$$

Combining this with Dolbeault's theorem, we can obtain a purely algebraic formulation of the Dolbeault cohomology groups,

$$H^{p,q}(X) = H^q(X, \Omega^p),$$

where Ω is the sheaf of (algebraic) differential forms on X. This gives us

$$H^i(X, \mathbb{C}) = \bigoplus_{p+q=i} H^q(X, \Omega^p),$$

and thus we have an honest-to-goodness algebraic method of computing the cohomology groups (albeit without torsion) for complex projective varieties.

Unfortunately, this procedure breaks down when we leave the safe haven of characteristic zero. First, it is not clear how to obtain cohomology with coefficients in a field of characteristic zero for such spaces. This is necessary, for example, if one wishes to use the Lefschetz fixed-point formula to count fixed points, as one must do to prove the Weil conjectures. Moreover, it can be shown by other means that the above method does not even provide the correct Betti numbers in characteristic p. It is clear, then, that any proper topological cohomology theory for general algebraic varieties (or schemes) will have to come from a different approach.

This new approach, developed by Grothendieck, required a reworking of the very notion of a topology. Yet it has its roots in the study of the most basic invariant of algebraic topology, the fundamental group. The fundamental group of a topological space X can be described in two ways. The first is as the group of based homotopy classes of maps $S^1 \to X$, or as the group of based loops in X. These notions clearly have no algebro-geometric counterpart, as neither the notion of S^1 nor that of a loop makes any sense, and the idea of a homotopy class is only slightly more sensible. The second description is as the automorphism group of the universal cover $\tilde{X} \to X$. Again, we are stuck with an object that does not exist in algebraic geometry, as algebraic varieties will not necessarily have universal covers. (Think of $X = \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$; analytically, the universal covering map is given by $\exp : X \to X$, which is a transcendental function.) But there are algebraic analogues of finite covering spaces, and this is where we will begin our study.

2. Étale Morphisms

We wish to develop the correct algebraic analogue of a topological covering space. Our motivation will come from the theory of smooth manifolds, where we have the inverse mapping theorem.

Theorem 2.1 (Inverse Mapping Theorem)**.** *Let $f : M \to N$ be a smooth map of smooth manifolds of equal dimension. Then f is locally an isomorphism at a point $x \in M$ if and only if $df_x : T_xM \to T_{f(x)}M$ is an isomorphism.*

Of course, even when the manifolds involved are smooth varieties over $\mathbb{C}$ and f is a morphism of varieties, this theorem may fail to hold in the Zariski topology. For example, take $M = N = \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$, $f : M \to N$ given by $z \mapsto z^2$. In the analytic setting, we can obtain a local inverse at a point $z_0 \in \mathbb{C}^\times$ by taking U to be $\mathbb{C}^\times$ minus a ray from the origin to infinity not intersecting z_0, and $f^{-1} : U \to \mathbb{C}^\times$ a branch of the square root function. But no such set U is open in the Zariski topology, so this inverse cannot be defined. Nevertheless, we will see that the hypothesis of the inverse mapping theorem is still a "good" notion of a covering map, and once we develop the basic notions of an étale topology, we will have an analogue of the inverse mapping theorem. Thus, we make the following definition.

Definition 2.2. Let X and Y be smooth varieties over an algebraically closed field k. A morphism $f : X \to Y$ is **étale** at a point $x \in X$ if $df_x : T_xX \to T_{f(x)}Y$ is an isomorphism.

We could apply this definition to non-smooth varieties, but it would not be a very useful notion, because the tangent space does not capture much information about a variety at a singular point. Instead, we require that f induce an isomorphism on tangent cones. If we assume our varieties are locally noetherian, this is equivalent to f being an isomorphism on completed local rings. If k is not algebraically closed, we say a map $f : X \to Y$ is étale at $x \in X$ if the corresponding map $f_{\overline{k}} : X_{\overline{k}} \to Y_{\overline{k}}$ is étale at every $x' \in X_{\overline{k}}$ lying over $x \in X$.

We'd like to replace this definition of étale with one that's more abstract, but more useful for proving properties. This definition will work for arbitrary schemes X and Y. To kill the suspense, I'll first give the definition of an étale morphism. Then, I'll explain the various components of the definition, and give some examples. Finally, I'll prove that it's equivalent to our definition for (locally Noetherian) varieties.

Definition 2.3. Let X and Y be schemes, $f : X \to Y$ a morphism of schemes. Then f is **étale** if it is flat and unramified. We say a ring homomorphism $\phi : A \to B$ is **étale** if the corresponding morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is étale.

Let's investigate what this definition means.

Definition 2.4. Let A and B be rings, $\phi : A \to B$ be a homomorphism of rings. Then ϕ is **flat** if the functor $B \otimes_A -$ is exact.

Definition 2.5. Let X and Y be schemes, $f : X \to Y$ a morphism of schemes. Then f is **flat** if the map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ of local rings is flat for all $x \in X$.

In fact, it suffices to check this property only at the closed points of X, as a homomorphism of rings is flat if and only if the corresponding local homomorphism is flat at all maximal ideals. (See Matsumura, Theorem 7.1.)

Flatness is a somewhat mysterious notion. A flat morphism $f : X \to Y$ is the algebro-geometric analogue of a continuous family of manifolds $X_y = f^{-1}(y)$ parametrized by the points of Y. Another way that this is often put is that the fibres of f "vary nicely." For example, we have the following basic result.

Proposition 2.6. *Let $f : X \to Y$ be a flat morphism of varieties over a field k. Then for all closed points $y \in Y$, the fibre $X_y = f^{-1}(y)$ satisfies*

$$\dim_x X_y = \dim_x X - \dim_y Y$$

for any closed point $x \in X_y$.

Proof. We induct on the dimension of Y. If $\dim Y = 0$, $X_y = X$, and so there is nothing to prove. Now suppose $\dim Y > 0$. Then $\exists t \in \mathfrak{m}_y \subseteq \mathcal{O}_{Y,y}$ such that t is not a zero divisor. Let $Y' = \operatorname{Spec} \mathcal{O}_{Y,y}/(t)$. By Krull's PID theorem, $\dim Y' = \dim Y - 1$. Let $X' = X \times_Y Y'$. Since t is not a zero divisor, the map $a \mapsto ta$ in $\mathcal{O}_{Y,y}$ is injective. Since f is a flat morphism, the map $b \mapsto f^\sharp(t)b$ in $\mathcal{O}_{X,x}$ must be injective, and hence $f^\sharp(t)$ is not a zero divisor. So again by Krull, $X' = \operatorname{Spec} \mathcal{O}_{X,x}/(f^\sharp(t))$ has dimension $\dim X - 1$. The base change $Y' \to Y$ does not change the fibre X_y over y, so we are done by induction. □

In fact, this property characterizes flat maps between nonsingular varieties.

A similar result in this vein is that if $f : X \to Y$ is a finite morphism, then flatness of f implies that the number of points in each fibre (counted with multiplicities) is constant. As a general rule, open inclusions are flat, while closed inclusions are not (unless it is also an open inclusion).

Definition 2.7. Let A and B be local rings, $\phi : A \to B$ be a homomorphism of local rings. Then ϕ is **unramified** if $B/f(\mathfrak{m}_A)B$ is a finite, separable field extension of $A/\mathfrak{m}_A$.

Note that this definition clearly requires $f(\mathfrak{m}_A)B = \mathfrak{m}_B$.

Definition 2.8. Let X and Y be schemes, $f : X \to Y$ a morphism of schemes. Then f is **unramified** if for every $x \in X$, the maps $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ are unramified, and furthermore, f is of finite type.[1]

This notion corresponds, for instance, to unramified coverings of Riemann surfaces and unramified extensions in number theory.

Example 2.9. Let $f : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ correspond to the map $\phi : t \mapsto t^2$ of the coordinate ring $A = B = \mathbb{C}[t]$. f sends the closed point $\mathfrak{p} = (t - p)$ of $\mathbb{A}^1_{\mathbb{C}}$ to $\mathfrak{p}' = (t - p^2)$. If

[1]Arguably, the correct definition should have "of finite type" replaced by "locally of finite presentation," but this distinction should not be important for us.

$p \neq 0$, the map on local rings satisfies

$$\begin{aligned}\phi(\mathfrak{p}')B_{\mathfrak{p}} &= (t^2 - p^2)B_{\mathfrak{p}} \\ &= (t-p)(t+p)B_{\mathfrak{p}} \\ &= (t-p)B_{\mathfrak{p}} \\ &= \mathfrak{p}.\end{aligned}$$

Thus f is unramified at the nonzero points. However, if $\mathfrak{p} = (t)$, then $f(\mathfrak{p}) = \mathfrak{p}$, and

$$\phi(\mathfrak{p})B_{\mathfrak{p}} = (t^2)B_{\mathfrak{p}} \neq \mathfrak{p},$$

so f is not unramified at 0.

The above example provides us with a flat morphism that is not unramified, and hence not étale. It is clear that a closed immersion of a subscheme into a variety is unramified, but not flat, hence not étale. Any open immersion is clearly flat and unramified, hence étale.

Example 2.10. Let k be a field. We wish to classify étale k-algebras A. For any prime $\mathfrak{p} \in \operatorname{Spec} A$, the local k-algebra $A_{\mathfrak{p}}$ is unramified over k, and hence it is a finite, separable field extension of k. This shows that A has dimension 0, as its localization at every prime is a field. A is a finitely generated k-algebra, hence Noetherian, so A is Artinian (Atiyah-MacDonald, Theorem 8.5). Every Artinian ring is a finite direct product of its local rings (*ibid.*, Theorem 8.7), so A is a finite direct product of finite separable field extensions. Conversely, a finite direct product of separable field extensions is clearly an étale k-algebra.

Example 2.11. Let A be a Dedekind domain with field of fractions K, and L a finite, separable extension of K. Let B be the integral closure of A in L, and let $\mathfrak{P}$ be a prime ideal of B. Then $\mathfrak{p} = \mathfrak{P} \cap A$ is a prime ideal of A. Then the following are easily seen to be equivalent.

(a) The map $A_{\mathfrak{p}} \to B_{\mathfrak{P}}$ is unramified.
(b) In the factorization of $\mathfrak{p}B$ into a product of prime ideals, $\mathfrak{P}$ occurs with exponent one, and the extension $B/\mathfrak{P} \supseteq A/\mathfrak{p}$ is separable.

Thus, our definition of "unramified" agrees with the equivalent notion in algebraic number theory. We claim that if $b \in B$ is contained in every prime ideal that ramifies, then the localization B_b is an étale A-algebra, and that conversely, any étale A-algebra will be a finite product of algebras of this type.

Example 2.12. If $X = \operatorname{Spec} A$ is connected and normal, K the field of fractions of A, and L a finite separable extension of K. Then the normalization of X in L is $\operatorname{Spec} B$, where B is the integral closure of A in L. If U is an open subset of $\operatorname{Spec} B$ that does not contain any closed points where the map $Y \to X$ is ramified, then $U \to X$ is flat, and hence étale. Any étale X-scheme is a disjoint union of schemes of this type.

We now give an important local form for an étale morphism, called a *standard étale* morphism. If A is a ring, $f(T) \in A[T]$ monic, then $B = A[T]/(f(T))$ is a free A-module of finite rank. If $b \in B$ such that $f'(T)$ is invertible in B_b, then $A \to B_b$ is étale. For example, we could take $A = \mathbb{R}[x]$, $f(T) = T^2 - xT + 3$. Then $f'(T)$ is invertible except where the curve $2T = x$ intersects the curve $T^2 - xT + 3 = 0$, so the map from the curve cut out by $T^2 - xT + 3$ to the x-axis is étale away from these two points.

Any morphism $f : X \to T$ that is isomorphic to the Spec of a morphism of the above type is called **standard étale**. If $f : X \to Y$ is any étale morphism, then for every $x \in X$, there exists affine open neighborhoods U of x and V of $f(x)$ such that $f|_U$ is standard étale. This gives a very clear geometric picture of what an étale morphism looks like.

References

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ÉTALE COHOMOLOGY SEMINAR LECTURE 2

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1. Étale Morphisms (Continued)

While étale morphisms do not satisfy the inverse mapping theorem (at least not in a form that we are ready to state yet), they share many good properties with local isomorphisms. We list a few here. For full proofs, see, for example, Milne's *Étale Cohomology*.

Proposition 1.1. (a) *Any open immersion is étale.*
(b) *The composite of two étale morphisms is étale.*
(c) *Any base change of an étale morphism is étale.*
(d) *If $\phi \circ \psi$ and ϕ are étale, then so is ψ.*

Proposition 1.2. *Let $f : X \to Y$ be an étale morphism.*
(a) *For all $x \in X$, $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,f(x)}$ have the same Krull dimension.*
(b) *The morphism f is quasi-finite.*
(c) *The morphism f is open.*
(d) *If Y is reduced, then so is X.*
(e) *If Y is normal, then so is X.*
(f) *If Y is regular, then so is X.*

Proposition 1.3. *Let $f : X \to Y$ be a morphism of finite type. The set of points $U \subset X$ where f is étale is open in X.*

Proposition 1.4. *Let $f : X \to Y$ be an étale morphism of varieties. If Y is connected, then any section of f is an isomorphism of Y with a connected component of X.*

Corollary 1.5. *Let f, f' be étale morphisms $X \to Y$ where X and Y are varieties over an algebraically closed field and X is connected. If f and f' agree at a single point of X, then they are equal on all of X.*

Finally, we prove that our two definitions of étale are equivalent in the case of varieties over an algebraically closed field.

Theorem 1.6. *Let $f : X \to Y$ be a morphism of varieties over an algebraically closed field. Then f is étale if and only if the induced map on completed local rings is an isomorphism at every point.*

Proof. Every morphism of varieties is of finite type, so it suffices to show that for every local homomorphism $\phi : A \to B$ arising from a morphism of k-varieties, the homomorphism $\widehat{\phi} : \widehat{A} \to \widehat{B}$ is an isomorphism if and only if ϕ is flat and unramified.

If $\widehat{\phi}$ is an isomorphism, then clearly $\widehat{\phi}$ is flat and unramified.

If $\widehat{\phi}$ is flat, then any exact sequence $M' \to M \to M''$ of A-modules gives an exact sequence

$$\widehat{B} \otimes_A M' \to \widehat{B} \otimes_A M \to \widehat{B} \otimes_A M''$$

because $\widehat{A}$ is a flat A-algebra (see Atiyah-MacDonald, 10.14). Since $\widehat{B}$ is a faithfully flat B-algebra (*ibid.*, 10, Ex. 7),

$$B \otimes_A M' \to B \otimes_A M \to B \otimes_A M''$$

is exact, and hence ϕ is flat.

If $\widehat{\phi}$ is unramified, then $\mathfrak{m}_B$ and $\mathfrak{m}_A B$ both generate the maximal ideal in $\widehat{B}$. If $R \to S$ is faithfully flat, and $\mathfrak{a}$ is an ideal of R, then $\mathfrak{a}S \cap R = \mathfrak{a}$ (see Matsumura, 4.C). Thus $\mathfrak{m}_B = \mathfrak{m}_A B$, and so ϕ is unramified. Thus, we have shown that if $\widehat{\phi}$ is an isomorphism, then ϕ is étale.

See Milne's lecture notes for a proof of the other direction. □

2. The Étale Fundamental Group

Before developing the theory of the étale fundamental group, we review the classical theory.

Let X be a connected, path-connected, semi-locally simply connected space, and fix a basepoint $x_0 \in X$. Then we define $\pi_1(X, x_0)$ to be the group of (based) homotopy classes of loops in X based at x_0. This description, as previously mentioned, is completely unsuitable for the setting of algebraic geometry, so we give an alternate definition.

A space X' equipped with a continuous map $\pi : X' \to X$ is called a **covering space** of X if, for every point $x \in X$, there exists a neighborhood U of x such that $\pi^{-1}(U)$ is the disjoint union of some collection of open sets $\{U_i \subseteq X'\}$, and such that $\pi|_{U_i}$ is a homeomorphism from U_i to U for all i. (Note that in the case of smooth manifolds, a surjective étale map is the same thing as a smooth covering space.) A map of covering spaces is simply a map $X' \to X''$ over X.

We call a covering space $\tilde{\pi} : \tilde{X} \to X$ equipped with a basepoint $\tilde{x}_0$ a **universal cover** if, for every covering space $\pi : X' \to X$ and point $x_0' \in X'$ over $x_0 \in X$, there is a unique covering space map $\tilde{X} \to X'$ sending $\tilde{x}_0'$ to x_0'. It is clear that the universal cover, if it exists, is unique up to unique isomorphism.

Proposition 2.1. *Under the hypotheses on* (X, x_0)*, the universal cover exists and is simply connected.*

Proof. Let $\tilde{X}$ be the space of endpoint-preserving homotopy classes of paths in X based at x_0. The trivial path is the basepoint of $\tilde{X}$, and $\tilde{X}$ is topologized by the quotient topology, where the space of based paths in X is given the compact open topology. The map $\tilde{\pi} : \tilde{X} \to X$ is given by sending a path ϕ to $\phi(1)$.

We omit the proof that the above construction gives rise to a simply connected covering space, and that it is the universal cover.

□

The universal cover gives us another interpretation of the fundamental group as follows. We write $\mathrm{Aut}_X(\tilde{X})$ for the group of covering space maps $\tilde{X} \to \tilde{X}$. If $\alpha \in \mathrm{Aut}_X(\tilde{X})$, then any path from $\tilde{x}_0$ to $\alpha\tilde{x}_0$ maps by π to a loop in X. Since $\tilde{X}$ is simply connected, the homotopy class of this loop is independent of the choice of path, and so we get a map $\mathrm{Aut}_X(\tilde{X}) \to \pi_1(X, x_0)$.

Proposition 2.2. *The map* $\mathrm{Aut}_X(\tilde{X}) \to \pi_1(X, x_0)$ *is an isomorphism.*

We have still not developed the theory well enough to apply to the algebro-geometric case: as we remarked earlier, topological spaces do not have universal covers. But there is another way to describe the universal cover. We let $\mathrm{Cov}(X)$ be the category whose objects covering spaces of X with finitely many connected components, and whose morphisms are covering maps. We define a functor $F : \mathrm{Cov}(X) \to \mathrm{Set}$ that sends a covering space $\pi : X' \to X$ to the set $\pi^{-1}(x_0)$. Then F is representable by $\tilde{X}$, as a covering space map $\tilde{X} \to X'$ is given by choosing a point of $\pi^{-1}(x_0)$. The action of $\pi_1(X, x_0)$ on $\tilde{X}$ gives an action on $\mathrm{Hom}_X(\tilde{X}, X')$.

Proposition 2.3. *The functor F gives an equivalence between* $\mathrm{Cov}(X)$ *and the category of $\pi_1(X, x_0)$-sets with finitely many orbits.*

In particular, we have the following Galois-like correspondence: connected covering spaces correspond to subgroups of $\pi_1(X, x_0)$ by associating a connected covering space X' with the stabilizer of a point of the $\pi_1(X, x_0)$-set $F(X')$. "Smaller" subgroups correspond to "larger" covering spaces, and vice-versa. (In fact, we will see in the étale setting that the usual Galois correspondence follows from the parallel statement for the étale fundamental group!)

Now, we are ready to formulate the theory for X an arbitrary connected scheme. We take, as our basepoint, a geometric point $\overline{x}_0 \to X$, where $\overline{x}_0 = \mathrm{Spec}\, k$, k separably algebraically closed. If X is a variety over an algebraically closed field k, this is the same as choosing a closed point x_0 of X.

Our replacement for the category $\mathrm{Cov}(X)$ will be the category FEt/X of finite, étale maps $\pi : X' \to X$. Étale morphisms are open, and finite morphisms are closed, so such morphisms are automatically surjective. In this setting, we have a functor $F : \mathrm{FEt}/X \to \mathrm{Set}$ sending (X', π) to the set of k-valued points of X' that lie over the point x_0. In the case of a variety over an algebraically closed field, $F(X') = \pi^{-1}(x_0)$.

Unlike in the topological setting, this functor F is not representable because there does not exist a universal cover, in general. However, Grothendieck showed that F is *pro-representable*, i.e., there is a projective system $\tilde{X} = (X_i)_{i \in I}$ of finite étale coverings X for some directed set I such that

$$F(X') = \mathrm{Hom}(\tilde{X}, X') = \varinjlim \mathrm{Hom}(X_i, X'), \quad \text{functorially in } X'.$$

This projective system $\tilde{X}$ will play the role of the universal cover in our theory. We first prove that such a $\tilde{X}$ in fact exists.

Lemma 2.4. *F is left-exact, in the sense that it preserves finite limits.*

Proof. It suffices to show that F commutes with fibre products. But this is immediate from the definition of F as the fibre over the basepoint. □

Lemma 2.5. *If $\phi : X' \to X''$ is a morphism in* FEt/X*, then ϕ is an isomorphism if and only if $F(\phi) : F(X') \to F(X'')$ is an isomorphism.*

Proof. One direction is clear. If $F(X') \to F(X'')$ is an isomorphism, then at the points $F(X'')$, the fibre of ϕ has one point. By 1.1(d), ϕ is étale. The number of points in the fibre of a finite flat map is locally constant, so since $F(X'')$ intersects every connected component of X'', ϕ is degree 1 everywhere. Hence, it is an isomorphism. □

Corollary 2.6. *If $\phi : X' \to X''$ is a morphism in* FEt/X*, then ϕ is injective if and only if $F(\phi) : F(X') \to F(X'')$ is injective.*

Proof. Injectivity of ϕ is equivalent to saying that the projection map $X'' \times_{X'} X'' \to X''$ is an isomorphism. The claim then follows from the previous two lemmas. $\square$

Lemma 2.7. *The category* FEt/X *is Artinian.*

Proof. By the corollary above, decreasing chain $X' \hookleftarrow X'' \hookleftarrow \cdots$ will then correspond to a decreasing chain $F(X') \hookleftarrow F(X'') \hookleftarrow \cdots$. Since the category of finite sets is Artinian, this chain must stabilize. By the previous lemma, this implies that the original chain stabilizes. Thus FEt/X is Artinian. $\square$

Lemma 2.8. *For every* $X'' \in \mathrm{FEt}/X$, $x'' \in F(X'')$, *there exists a minimal element* $X' \in \mathrm{FEt}/X$ *and a point* $x' \in F(X')$, *and a map* $\phi : X' \to X''$ *with* $F(\phi)(x') = x''$, *and this map is uniquely determined by* X', X'', x', *and* x''.

Proof. The existence of (X', x') and ϕ follows immediately from the previous lemma; choose a minimal object among those with such maps to X''. To see uniqueness, suppose $\phi_1, \phi_2 : X' \to X''$ satisfy $F(\phi_i)(x') = x''$. Let Y be the equalizer of ϕ_1 and ϕ_2. Then by left-exactness of F, $F(Y)$ is the equalizer of $F(\phi_1)$ and $F(\phi_2)$. This is nonempty because x' is in this equalizer, so $F(Y)$ is a nonempty subset of $F(X')$. But this implies that $Y \subseteq X'$, and so $Y = X'$ by minimality. Thus $\phi_1 = \phi_2$. $\square$

Finally, we are ready to prove our theorem.

Theorem 2.9. *The functor* F *is pro-representable.*

Proof. We construct our projective system $\tilde{X} = (X_i)_{i \in I}$ by letting $\{X_i\}$ be the set of all minimal objects equipped with distinguished basepoints x_i, and the maps $X_j \to X_i$ given by surjective basepoint preserving morphisms. For $X' \in \mathrm{FEt}/X$, we define $\tilde{F}(X') = \varinjlim \mathrm{Hom}(X_i, X')$. We get a natural (functorial) map $\tilde{F}(X') \to F(X')$ via $\phi \mapsto \phi(x_i)$ for some $\phi_i \in \mathrm{Hom}(X_i, X')$ representing ϕ. We claim that this map is an isomorphism. Surjectivity is immediate from the previous lemma. To see that it is injective, suppose $\phi_i : X_i \to X'$ and $\phi_j : X_j \to X'$ get mapped to some $x' \in F(X')$. Then the fibre product $X_i \times_{X'} X_j$ is minimal and lies over X_i and X_j, and the induced map $\phi_i \times \phi_j : X_i \times_{X'} X_j$ represents both ϕ_i and ϕ_j, and thus $\phi_i = \phi_j$ in $\tilde{F}(X')$. So we are done.

$\square$

We would like to define our étale fundamental group as $\mathrm{Hom}(\tilde{X}, \tilde{X})$, but it is unclear what this means. It should evidently be some sort of limit of the sets $\mathrm{Hom}(X_i, X_i)$. We might hope that $\mathrm{Hom}(X_i, X_i)$ is the same as $\mathrm{Hom}(\tilde{X}, X_i)$, which would allow us to take a projective limit, but this is not always the case. But it turns out that covers X_i for which the natural injection $\mathrm{Hom}(X_i, X_i) \to \mathrm{Hom}(\tilde{X}, X_i) = F(X_i)$ is an isomorphism are cofinal in $\tilde{X}$, and so we may take $\tilde{X}$ to consist precisely of these covers. Such a cover is called a **Galois cover**. The following equivalent characterizations are evident.

Proposition 2.10. *The following are equivalent.*

(a) X_i *is Galois.*
(b) $\mathrm{Aut}_X(X_i)$ *acts transitively on* $F(X_i)$.
(c) $\mathrm{Aut}_X(X_i)$ *acts simply transitively on* $F(X_i)$.

For X_i and X_j Galois, the map $\phi_{ji} : X_j \to X_i$ induces a surjective homomorphism $(\phi_{ji})* : \mathrm{Aut}_X(X_j) \to \mathrm{Aut}_X(X_i)$ via the identifications

$$\mathrm{Aut}_X(X_\ell) = \mathrm{Hom}(X_\ell, X_\ell) \cong \mathrm{Hom}(\tilde{X}, X_\ell) = F(X_\ell).$$

We now define the **étale fundamental group** to be the profinite group

$$\pi_1(X, \overline{x}_0) = \mathrm{Hom}(\tilde{X}, \tilde{X}) = \varprojlim \mathrm{Aut}_X(X_i).$$

Example 2.11. Let k be an algebraically closed field of characteristic zero, $X = \mathbb{A}^1_k \setminus \{0\}$, $\overline{x}_0$ arbitrary. The minimal (i.e., connected) finite étale covers of X are the maps $X_n = X \to X$, $t \mapsto t^n$. These are all Galois, with $\mathrm{Aut}_X(X_n) = \mu_n(k)$, the group of nth roots of unity in k. Thus

$$\pi_1(X, \overline{x}_0) = \varprojlim \mu_n(k) \cong \widehat{\mathbb{Z}},$$

the profinite completion of $\mathbb{Z}$.

We note that $\widehat{\mathbb{Z}}$, via its right action on $\tilde{X}$, will act on $F(X')$ on the left for any finite étale cover $X' \to X$. There will be some X_i with covering maps $X_i \to X'$ that surject onto each connected component of X'. Thus, the action of $\widehat{\mathbb{Z}}$ on $F(X')$ factors through the finite quotient $\mathrm{Aut}_X(X_i)$. This proves that the action is continuous.

Theorem 2.12. *The functor $X' \mapsto F(X')$ is an equivalence between* FEt/X *and the category of finite discrete $\pi_1(X, \overline{x}_0)$-sets.*

Example 2.13. Let k be an algebraically closed field of characteristic zero, $X = \mathbb{A}^1_k$. If $\pi : X' \to \mathbb{A}^1_k$ is an étale covering of degree n, it will extend to a covering $\overline{\pi} : \overline{X'} \to \mathbb{P}^1_k$ ramified at ∞ with ramification index n. If we take $\omega = dz$ on $\mathbb{P}^1_k$, then $\overline{\pi}^{-1}(\omega)$ will have no zeroes and at most $2n - (n-1) = n+1$ poles. Then $-2 \le 2g_{\overline{X'}} - 2 \le -(n+1)$, and thus $n = 1$. Thus $\overline{\pi}$, and hence π, is an isomorphism. This proves that $\pi_1(\mathbb{A}^1_k, \overline{x}_0) = 1$ for any point $\overline{x}_0$.

If char $k = p > 0$, then there exist nontrivial étale coverings of $\mathbb{A}^1$. For example, the map $x \mapsto x^p + x$ defines a nontrivial étale covering $\mathbb{A}^1 \to \mathbb{A}^1$. This map is étale because its derivative is $px^{p-1} + 1 = 1$.

There is an important comparison theorem for varieties over $\mathbb{C}$. Let X be a nonsingular variety over $\mathbb{C}$, $\pi : X' \to X$ a finite étale covering. Then X' is nonsingular, and endowing $X'(\mathbb{C})$ and $X(\mathbb{C})$ with their complex topologies makes $\pi_{\mathbb{C}} : X'(\mathbb{C}) \to X(\mathbb{C})$ a finite covering space.

Theorem 2.14 (Riemann Existence Theorem)**.** *The functor sending (X', π) to $(X'(\mathbb{C}), \pi_{\mathbb{C}})$ is an equivalence of categories between* FEt/X *and the category of finite covering spaces of $X(\mathbb{C})$.*

As a corollary, any finite covering space of $X(\mathbb{C})$ can be realized as a quotient of some $X_i(\mathbb{C})$. If $x_0 \in X(\mathbb{C})$, then

$$\pi_1(X, \overline{x}_0) = \varprojlim \mathrm{Aut}_X(X_i) = \varprojlim \mathrm{Aut}_{X(\mathbb{C})}(X_i(\mathbb{C})) = \pi_1(X(\mathbb{C}), x_0)\hat{}.$$

In other words, the étale fundamental group of a nonsingular complex variety is the profinite completion of the usual fundamental group. This agrees with our earlier calculations.

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ÉTALE COHOMOLOGY SEMINAR LECTURE 4

EVAN JENKINS

I am grateful to Thanos Papaioannou for presenting most of the material in this lecture to me. This presentation is based on SGA 1, Exposé VIII.

1. Sheaves on the Étale and fpqc Sites

In general, the sheaf criterion on the étale topology may be difficult to verify directly, as a scheme will in general have many étale covers. It is clear that a necessary condition for a presheaf $\mathcal{F}$ to be a sheaf on X_{et} is that it be a sheaf with respect to Zariski covers (i.e., its restriction to X_{zar} is a sheaf), and that it be a sheaf with respect to one-piece étale covers $(V \to U)$ such that V and U are affine. We have formulated this awkward necessary condition because, in fact, it is also sufficient.

To illustrate the flexibility of this result will show this in a slightly more general setting, the **fpqc topology**. The objects of the site X_{fpqc} are maps $U \to X$ that are flat and locally quasicompact, and covers are flat, locally quasicompact, and jointly surjective families of morphisms.[1] The proof we give will immediately imply the result for the étale site.

Theorem 1.1. *Let $\mathcal{F} : X_{\mathrm{fpqc}}^{\mathrm{op}} \to \mathrm{Set}$ be a presheaf such that $\mathcal{F}|_{X_{\mathrm{zar}}}$ is a sheaf, and for every fpqc cover $(V \to U)$ with V and U affine, the diagram*

$$\mathcal{F}(U) \to \mathcal{F}(V) \rightrightarrows \mathcal{F}(V \times_U V)$$

is exact. Then $\mathcal{F}$ is a sheaf.

We note that this theorem is a special case of a more general result about stacks; namely, if $\mathcal{F}$ is a fibred category over the fpqc site, then $\mathcal{F}$ is a stack if it is a Zariski stack and a stack with respect to one-piece affine fpqc covers.

Proof. Let $(V_i \xrightarrow{f_i} U)_{i \in I}$ be an fpqc cover. Set $V = \coprod_{i \in I} V_i$, with the obvious map $f : V \to U$. Then $(V \to U)$ is a one-piece fpqc cover. We have the following diagram, where vertical arrows are given by the obvious restriction maps.

(1.1)
$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] & \prod_{i \in I} \mathcal{F}(V_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{(i,j) \in I \times I} \mathcal{F}(V_i \times_U V_j) \\
\mathcal{F}(U) \arrow[r] \arrow[u, equal] & \mathcal{F}(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[u] & \mathcal{F}(V \times_V V) \arrow[u]
\end{tikzcd}$$

[1]The original (and perhaps "usual") definition of fpqc does not include the word "locally." However, étale maps (and even inclusions of Zariski open sets) can fail to be quasicompact, so we must replace quasicompactness by local quasicompactness if we wish the fpqc topology to be strictly stronger than the Zariski and étale topologies.

We claim that the vertical arrows are isomorphisms, so that exactness of the top line will reduce to exactness of the bottom line. This follows from the following easy fact.

Lemma 1.2. *If $\varnothing$ is the empty scheme (with the empty map to X), then $\mathcal{F}(\varnothing)$ is a one-point set.*

Proof. The scheme $\varnothing$ has the empty cover (i.e., the cover containing no schemes) in the Zariski topology. Since $\mathcal{F}$ is a sheaf when restricted to the Zariski topology, it follows from the sheaf condition that $\mathcal{F}(\varnothing)$ is equal to the empty product of sets, which is a one-point set. □

Since the V_i form a Zariski cover of V, the first vertical arrow of (1.1) fits into an exact sequence

$$\begin{tikzcd}
\prod_{(i,j)\in I\times I}\mathcal{F}(V_i\times_V V_j) \\
\prod_{i\in I}\mathcal{F}(V_i) \arrow[u, shift left] \arrow[u, shift right] \\
\mathcal{F}(U) \arrow[u]
\end{tikzcd}$$

But by the lemma, $\mathcal{F}(V_i\times_V V_j)$ is a one-point set if $i\neq j$, and so the images of the two vertical maps automatically agree. Hence, the bottom arrow is an isomorphism. Similarly, $(V_i\times_U V_j)_{(i,j)\in I\times I}$ forms a disjoint Zariski cover of $V\times_U V$, so the same argument shows that the second vertical arrow in (1.1) is an isomorphism.

This shows that exactness of the top row of (1.1) is equivalent to exactness of the bottom row. In particular, if V_i is a finite affine cover, V is affine, so by hypothesis, the bottom row is exact.

We have reduced the problem to showing that $\mathcal{F}$ satisfies the sheaf condition on our one-piece (not necessarily affine) fpqc cover $(V\xrightarrow{f}U)$. Since f is fpqc, V decomposes as $V=\bigcup_{i\in I}V_i$, where V_i are open, quasicompact subschemes such that their images $f(V_i)$ in U are affine open subschemes. Since the V_i are quasicompact, for each i we can write $V_i=\bigcup_{j\in J_i}V_{ij}$, where the V_{ij} are affine open subschemes of V_i, and J_i is a finite set. In the following diagram, the first two columns and middle row are easily seen to be exact by hypothesis. A diagram chase shows that the top row is exact, which is the desired result.

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \mathcal{F}(V\times_U V) \arrow[d] \\
\prod\limits_{i\in I,j\in J_i}\mathcal{F}(f(V_{ij})) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod\limits_{i\in I,j\in J_i}\mathcal{F}(V_{ij}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \prod\limits_{\substack{i\in I,i'\in I\\ j\in J_i,j'\in J_{i'}}}\mathcal{F}(V_{ij}\times_{f(V_{ij})}V_{i'j'}) \\
\prod\limits_{\substack{i\in I,i'\in I\\ j\in J_i,j'\in J_{i'}}}\mathcal{F}(f(V_{ij})\times_U f(V_{i'j'})) \arrow[r] & \prod\limits_{\substack{i\in I,i'\in I\\ j\in J_i,j'\in J_{i'}}}\mathcal{F}(V_{ij}\times_V V_{i'j'}) &
\end{tikzcd}$$

□

The proof is nearly identical if we replace the category Set with other typical categories.

Suppose $\mathcal{F}$ is a presheaf on X_{fpqc} (or X_{et}) that restricts to a Zariski sheaf. An example is the "structure sheaf" we defined last time. Our theorem reduces the problem of showing $\mathcal{F}$ is a sheaf to the problem of showing that it satisfies the sheaf condition for one-piece affine coverings. The following lemma is extremely useful in light of the fact that surjective flat maps of affine schemes correspond to *faithfully flat* morphisms of rings.

Lemma 1.3 (The Miraculous Section)**.** *Let $\phi : A \to B$ be a faithfully flat morphism of rings. Then the sequence*

$$0 \longrightarrow A \xrightarrow{\phi=\delta^0} B \xrightarrow{\delta^1} B \otimes_A B \xrightarrow{\delta^2} B \otimes_A B \otimes_A B \longrightarrow \cdots \tag{1.2}$$

is exact, where $\delta^k = \sum_{i=0}^{k} (-1)^i e_i$, $e_i(b_1 \otimes \cdots \otimes b_k) = b_1 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_k$.

Proof. We leave the verification that (1.2) is a complex as an exercise. To show that it is exact, it suffices to construct a homotopy between the identity map and the zero map.

First, we assume ϕ has a section $\sigma : B \to A$. Define a homotopy κ_i by $b_1 \otimes \cdots \otimes b_i \mapsto \sigma(b_1) b_2 \otimes b_3 \otimes \cdots \otimes b_i$.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\delta^0"] & B \arrow[r, "\delta^1"] \arrow[dl, "\kappa_1"] & B \otimes_A B \arrow[r, "\delta^2"] \arrow[dl, "\kappa_2"] & B \otimes_A B \otimes_A B \arrow[r] \arrow[dl, "\kappa_3"] & \cdots \\
0 \arrow[r] & A \arrow[r, "\delta^0"] & B \arrow[r, "\delta^1"] & B \otimes_A B \arrow[r, "\delta^2"] & B \otimes_A B \otimes_A B \arrow[r] & \cdots
\end{tikzcd}$$

To say that the identity is nullhomotopic via κ is to say that $\kappa_{i+1}\delta^i + \delta^{i-1}\kappa_i$ is the identity. We leave this straightforward verification as an exercise.

Now, we return to the general case. Since $A \to B$ is faithfully flat, it suffices to show that (1.2) is exact when tensored by B. In this case, we have a section σ given by multiplication.

$$\begin{tikzcd}
B \otimes_A B \arrow[dr, "\sigma"] & \\
A \otimes_A B \arrow[u, "\phi \otimes B"] \arrow[r, equal] & B
\end{tikzcd}$$

So we are done. □

Corollary 1.4. *The structure sheaf $\mathcal{O}_{X_{\text{fpqc}}}$, defined by $\mathcal{O}_{X_{\text{fpqc}}}(U) = \mathcal{O}_U(U)$, is in fact a sheaf.*

Proof. By definition, this presheaf is a Zariski sheaf, so it suffices by Theorem 1.1 to check the sheaf criterion for one-piece affine covers ($\operatorname{Spec} B \to \operatorname{Spec} A$). Such a cover corresponds to a faithfully flat map $\phi : A \to B$, so by Lemma 1.3, the sequence

$$0 \to A \to B \to B \otimes_A B$$

of A-modules is exact. But the last map, by definition, is simply the difference of the two restriction maps $B \to B \otimes_A B$, so exactness of this sequence implies the sheaf condition for ($\operatorname{Spec} B \to \operatorname{Spec} A$). Thus $\mathcal{O}_{X_{\text{fpqc}}}$ is a sheaf. □

Corollary 1.5. *Let $\mathcal{F}$ be a coherent sheaf on X in the Zariski sense. Then we can extend $\mathcal{F}$ to an fpqc presheaf by taking, for any fpqc map $f : U \to X$, $\mathcal{F}(U) = (f^*\mathcal{F})(U)$. This presheaf is a sheaf.*

Proof. As before, it suffices to check the sheaf condition for covers of the form $(\operatorname{Spec} B \to \operatorname{Spec} A)$, where $\phi : A \to B$ is faithfully flat. A coherent sheaf on $\operatorname{Spec} A$ corresponds to a finitely generated A-module M, while its pullback to $\operatorname{Spec} B$ will be $B \otimes_A M$. The proof of Lemma 1.3 holds in more generality; namely, if we replace A by $M = A \otimes_A M$, we get an exact sequence

$$0 \to M \to B \otimes_A M \to B \otimes_A B \otimes_A M \to \cdots ,$$

where the last map is the difference of the two restriction maps in the sheaf condition. Thus, the sheaf condition is satisfied. $\square$

Note that all of these proofs hold on the étale topology as well as the fpqc topology; it suffices to note that a flat morphism of rings $A \to B$ such that $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective is automatically faithfully flat.

Perhaps the most important class of sheaves on the fpqc (or étale) topology are those that arise from schemes. Recall that a functor $\mathcal{F} : X_{\mathrm{fpqc}}^{\mathrm{op}} \to \mathrm{Set}$ is called **representable** if $\mathcal{F}$ is naturally isomorphic to $\operatorname{Hom}(-, Z)$ for some X-scheme Z. We denote this functor by $\mathcal{Y}_Z$. We wish to show that any such functor is a sheaf.

We begin with the case that $Z = \operatorname{Spec} C$ is affine.

Lemma 1.6. *If $Z = \operatorname{Spec} C$ is affine, then $\mathcal{Y}_Z$ is an fpqc-sheaf.*

Proof. It is clear that such a presheaf is always a Zariski sheaf, so by Theorem 1.1, it suffices to check the sheaf condition for some fpqc cover $(\operatorname{Spec} B \to \operatorname{Spec} A)$ given by a faithfully flat morphism $\phi : A \to B$. By Lemma 1.3,

$$0 \to A \to B \to B \otimes_A B$$

is exact. Since $\operatorname{Hom}_{A-\mathrm{alg}}(C, -)$ is left-exact, it follows that

$$0 \to \operatorname{Hom}(C, A) \to \operatorname{Hom}(C, B) \to \operatorname{Hom}(C, B \otimes_A B)$$

is exact, and this is precisely the sheaf condition for $(\operatorname{Spec} B \to \operatorname{Spec} A)$. $\square$

Theorem 1.7. *For any X-scheme Z, $\mathcal{Y}_Z$ is an fpqc-sheaf.*

Proof. We write $Z = \bigcup_{i \in I} Z_i$, where Z_i are affine. It suffices to show that, by Theorem 1.1, the diagram

$$\mathcal{Y}_Z(U) \to \mathcal{Y}_Z(V) \rightrightarrows \mathcal{Y}_Z(V \times_U V) \tag{1.3}$$

is exact, where $(f : V \to U)$ is an affine cover.

Let $g, g' \in \operatorname{Hom}_X(U, Z)$ such that $g \circ f = g' \circ f$. Since f is surjective, g and g' must agree as topological maps. If we set $U_i = g^{-1}(Z_i) = g'^{-1}(Z_i)$, then U_i are open subspaces, and hence open subschemes of U. Applying the lemma to g and g' restricted to U_i, with images in the affine scheme Z_i, tells us that g and g' must agree as maps of schemes on U_i. The U_i cover U, so we must have $g = g'$. This shows that the first arrow in (1.3) is an injection.

Now suppose $g \in \mathrm{Hom}_X(U, Z)$ such that $g \circ p_1 = g \circ p_2$ as in the following diagram. We wish to show that g factors through a map g' as shown.

$$\begin{tikzcd} & V \times_U V \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \\ U \arrow[dr, "g'"'] & V \arrow[l, two heads, "f"] \arrow[d, "g"] \\ & Z \end{tikzcd}$$

For $u \in U$, choose a $v \in V$ such that $u = f(v)$. We claim that we can define g' topologically by setting $g'(u) = g(v)$. This would be well-defined if $V \times_U V$ were the topological fibre product, but it is not. However, the following lemma (whose proof we leave as an exercise) shows that $V \times_U V$ contains the points of the topological fibre product, so well-definedness follows.

Lemma 1.8. *If $f_1 : X_1 \to Y$, $f_2 : X_2 \to Y$ are morphisms of schemes, and $x_1 \in X_1$, $x_2 \in X_2$ such that $f_1(x_1) = f_2(x_2)$, then $\exists z \in X_1 \times_Y X_2$ such that $p_1(z) = x_1$, $p_2(z) = x_2$.*

All that is left to show is that g' comes from a map of schemes. Using the decomposition $Z = \bigcup_{i \in I} Z_i$, the previous lemma implies that we get maps $g_i'' : U_i \to Z_i$ such that the following diagram commutes.

$$\begin{tikzcd} & V_i \times_{U_i} V_i \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \\ U_i \arrow[dr, "g_i''"'] & V_i \arrow[l, two heads, "f"] \arrow[d, "g"] \\ & Z_i \end{tikzcd}$$

The scheme map g_i'' must agree with g' topologically on U_i. To see that the g_i'' glue together to form a scheme map g'' that agrees with g' on all of U, we simply cover the intersections of U_i and U_j with affines and we see that the maps must agree there. $\square$

Of particular importance will be those sheaves $\mathcal{Y}_Z$ for which Z is an affine group scheme. We typically write $Z(U)$ for $\mathcal{Y}_Z(U)$ in this case.

Example 1.9.
(a) Let μ_n be the affine X-scheme defined by $T^n - 1 = 0$. Then $\mu_n(U)$ is the group of nth roots of unity in $\Gamma(U, \mathcal{O}_U)$.
(b) Let $\mathbb{G}_a$ be the affine line over X as a group under addition. Then $\mathbb{G}_a(U)$ is the additive group of $\Gamma(\mathcal{U}, \mathcal{O}_U)$.
(c) Let $\mathbb{G}_m$ be the affine line over X minus the origin. Then $\mathbb{G}_m(U)$ is the multiplicative group of units $\Gamma(\mathcal{U}, \mathcal{O}_U)^\times$.
(d) Let GL_n be the affine X-scheme defined by $T \det(T_{ij}) = 1$ where $1 \leq i, j, \leq n$. Then $GL_n(U) = GL_n(\Gamma(U, \mathcal{O}_U))$.
(e) Let Λ be a finite group. We write $X \times \Lambda$ for the disjoint union of $|\Lambda|$ copies of X indexed by the group Λ. Then $\mathcal{Y}_{X \times \Lambda}$ is a constant sheaf of groups.

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ÉTALE COHOMOLOGY SEMINAR LECTURE 5

EVAN JENKINS

1. Sheaves on the Étale Site (Continued)

The following definition is an obvious generalization of our notion of a local ring for the étale topology.

Definition 1.1. Let X be a scheme, $\overline{x}$ a geometric point of X, and $\mathcal{F}$ a presheaf on X_{et}. The **stalk** $\mathcal{F}_{\overline{x}}$ of $\mathcal{F}$ at $\overline{x}$ is defined as $\mathcal{F}_{\overline{x}} = \varinjlim \mathcal{F}(U)$, where the limit runs over all étale neighborhoods $(U, \overline{u})$ of $\overline{x}$.

Example 1.2. Let k be a field. Recall that an étale k-algebra is a finite direct product of finite, separable extensions of k. Suppose $\mathcal{F}$ is a sheaf of abelian groups on $\operatorname{Spec} k$. The sheaf condition implies that if k''/k' is a finite Galois extension, where k' is finite and separable over k, then

$$\mathcal{F}(k') \to \mathcal{F}(k'') \rightrightarrows \mathcal{F}(k'' \otimes_{k'} k'')$$

is exact. It is a straightforward exercise to check that $k'' \otimes_{k'} k'' \cong (k'')^{[k'':k']}$ in such a way that the first inclusion of k'' is the identity on each factor, while the second inclusion twists by the action of a unique element of the Galois group on each factor. The sheaf condition also implies that products are sent to direct sums under $\mathcal{F}$, so it follows that $\mathcal{F}(k') \cong \mathcal{F}(k'')^{\operatorname{Gal}(k''/k')}$.

Since finite Galois extensions are cofinal among all extensions, it follows that the sheaf $\mathcal{F}$ is uniquely determined by its stalk over the geometric point k^{sep}. This identification determines an equivalence of categories between sheaves of abelian groups on $(\operatorname{Spec} k)_{\text{et}}$ and discrete $\operatorname{Gal}(k^{\text{sep}}/k)$-modules, where the forward direction sends a sheaf to its stalk over k^{sep}, and the reverse direction sends a discrete $\operatorname{Gal}(k^{\text{sep}}/k)$-module M to the sheaf $\mathcal{F}_M(A) = \operatorname{Hom}_{\operatorname{Gal}(k^{\text{sep}}/k)}(\operatorname{Hom}_{k-\text{alg}}(A, k^{\text{sep}}), M)$.

Definition 1.3. We call a sheaf $\mathcal{F}$ a **skyscraper sheaf** if $\mathcal{F}_{\overline{x}} = 0$ for any geometric point $\overline{x} \to X$ with image lying outside a finite set of points in X.

For Λ an abelian group, $\overline{x} \to X$ a geometric point, we define, for $U \to X$ étale,

$$\Lambda^{\overline{x}}(U) = \bigoplus_{\operatorname{Hom}_X(\overline{x}, U)} \Lambda.$$

If the image x of $\overline{x}$ is closed, then this is a skyscraper sheaf with support at x. To give a morphism $\mathcal{F} \to \Lambda^{\overline{x}}$ is to give compatible systems of morphisms $\mathcal{F}(U) \to \Lambda$ for every étale neighborhood $(U, \overline{u})$ of $\overline{x}$. This is equivalent to giving a morphism $\mathcal{F}_{\overline{x}} \to \Lambda$. Thus,

$$\operatorname{Hom}(\mathcal{F}, \Lambda^{\overline{x}}) \cong \operatorname{Hom}(\mathcal{F}_{\overline{x}}, \Lambda).$$

2. The Category of Sheaves

Let X be a scheme. We denote by $\mathrm{Sh}(X_{\mathrm{et}})$ the category of sheaves of abelian groups on X_{et}. A morphism of sheaves is simply a natural transformation of functors. This category is clearly additive with the obvious biproduct, $(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$. We will see that, as in the case of Zariski sheaves, this category is abelian.

Lemma 2.1. *Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X_{et}. Then the following are equivalent.*

(a) *α is surjective, i.e., $\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \to 0$ is exact.*
(b) *α is locally surjective, i.e., for every $U \in X_{\mathrm{et}}$, $s \in \mathcal{G}(U)$, there exists a covering $(U_i \to U)_{i \in I}$ such that for all $i \in I$, there exists $t_i \in \mathcal{F}(U_i)$ with $\alpha(t_i) = s|_{U_i}$.*
(c) *α is surjective on stalks, i.e., for every geometric point $\overline{x} \to X$, $\mathcal{F}_{\overline{x}} \xrightarrow{\alpha_{\overline{x}}} \mathcal{G}_{\overline{x}} \to 0$ is exact.*

Proof. (b) $\Rightarrow$ (a). Suppose $\beta : \mathcal{G} \to \mathcal{H}$ is a morphism of sheaves such that $\beta \circ \alpha = 0$. Let $s \in \mathcal{G}(U)$ for some étale $U \to X$. Then there is some étale cover $(U_i \to U)_{i \in I}$, $s_i \in \mathcal{F}(U_i)$, such that $\alpha(s_i) = s|_{U_i}$ for all $i \in I$. Thus, $\beta(s)|_{U_i} = (\beta \circ \alpha)(s_i) = 0$, and so $\beta(s) = 0$ by the sheaf condition on $\mathcal{H}$. Hence α is surjective.

(a) $\Rightarrow$ (c). Fix a geometric point $\overline{x} \to X$. Let $\Lambda = \operatorname{coker}(\mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}})$. We wish to show $\Lambda = 0$. The surjective cokernel map $\mathcal{G}_{\overline{x}} \to \Lambda$ corresponds to a map $\mathcal{G} \to \Lambda^{\overline{x}}$. But the composition $\mathcal{F} \to \mathcal{G} \to \Lambda^{\overline{x}}$ corresponds to the composition $\mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}} \to \Lambda$, which is the zero morphism. Since $\mathcal{F} \to \mathcal{G} \to 0$ is exact, this implies that our map $\mathcal{G} \to \Lambda^{\overline{x}}$ is the zero morphism, and hence the image of $\mathcal{G}_{\overline{x}}$ in Λ is zero. Since this map was surjective, we must have $\Lambda = 0$ as desired.

(c) $\Rightarrow$ (b). It is clear that if $U \to X$ is étale, $\overline{u} \to U$ a geometric point of U, then $\mathcal{F}_{\overline{u}} \cong \mathcal{F}_{\overline{x}}$, where $\overline{x}$ is the geometric point of $\overline{x}$ given by composition of $\overline{u}$ with $U \to X$. Thus, for every $\overline{u} \to U$, $\mathcal{F}_{\overline{u}} \to \mathcal{G}_{\overline{u}}$ is surjective. This means that for each $\overline{u} \to U$, there is some étale neighborhood $(V, \overline{v})$ of $\overline{u}$ with $s|_V$ in the image of $\mathcal{F}(V) \xrightarrow{\alpha|_V} \mathcal{G}(V)$. Taking the union of these V as a cover of U, we have that α is locally surjective. ☐

There is an obvious forgetful functor from $\mathrm{Sh}(X_{\mathrm{et}})$ to the category $\mathrm{PreSh}(X_{\mathrm{et}})$ of presheaves of abelian groups on X_{et}. Just as in the case of Zariski sheaves, this functor has a left adjoint, called *sheafification.* A simple construction of the sheafification of a presheaf $\mathcal{F}$ is given by $\mathcal{F}^{\mathrm{sh}} = \prod_{x \in X} (\mathcal{F}_{\overline{x}})^{\overline{x}}$, where $\overline{x}$ is any geometric point lying over x. It is straightforward to check that $\mathcal{F}^{\mathrm{sh}}$ satisfies the appropriate universal property.

Proposition 2.2. *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ be a sequence of sheaves of abelian groups on X_{et}. Then the following are equivalent.*

(a) *The sequence is exact.*
(b) *For every étale $U \to X$, the sequence $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$ is exact.*
(c) *For every geometric point $\overline{x} \to X$, the sequence $0 \to \mathcal{F}'_{\overline{x}} \to \mathcal{F}_{\overline{x}} \to \mathcal{F}''_{\overline{x}}$ is exact.*

Proof. Since the forgetful functor $\mathrm{Sh}(X_{\mathrm{et}}) \to \mathrm{PreSh}(X_{\mathrm{et}})$ admits a left adjoint, it is left exact. Condition (b) is precisely the condition of left exactness in the category of presheaves, so (a) and (b) are equivalent. Since taking direct limits preserves exactness, (b) implies (c).

Finally, suppose (c) holds, and $s' \in \mathcal{F}'(U)$ is mapped to zero in $\mathcal{F}(U)$. Then its image is zero on every stalk, so s' is zero at every stalk of $\mathcal{F}'$. Thus $s = 0$. Similarly, if $s \in \mathcal{F}(U)$ maps to zero in $\mathcal{F}''(U)$, then s must restrict to an element of $\mathcal{F}'_x$ on every stalk, from which it follows that $s \in \mathcal{F}'(U)$. Thus (c) implies (b). □

Corollary 2.3. *Let* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *be a sequence of sheaves of abelian groups on* X_{et}. *Then the following are equivalent.*

(a) *The sequence is exact.*
(b) *For every étale* $U \to X$*, the sequence* $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$ *is exact, and* $\mathcal{F} \to \mathcal{F}''$ *is locally surjective.*
(c) *For every geometric point* $\overline{x} \to X$*, the sequence* $0 \to \mathcal{F}'_{\overline{x}} \to \mathcal{F}_{\overline{x}} \to \mathcal{F}''_{\overline{x}} \to 0$ *is exact.*

Corollary 2.4. *The category* $\mathrm{Sh}(X_{\text{et}})$ *is abelian.*

Proof. The kernel of a morphism is the usual presheaf kernel, which is obviously a sheaf. The cokernel of a morphism is the sheafification of the presheaf cokernel. Images and coimages are isomorphic because they are isomorphic on stalks. □

We now give examples of two important exact sequences on the étale site. Note that these sequences are clearly *not* exact in the Zariski topology. This is our first real sign that étale cohomology may be more powerful than Zariski cohomology.

Example 2.5 (The Kummer sequence)**.** Let $n \in \mathbb{Z}$ be relatively prime to the characteristics of all residue fields of X. We define the **Kummer sequence** of degree n to be the sequence

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{t \mapsto t^n} \mathbb{G}_m \to 0.$$

We wish to show this sequence is exact. By Corollary 2.3, this is equivalent showing that the stalks are exact at every geometric point $\overline{x} \to X$. If $A = \mathcal{O}_{X,\overline{x}}$, then the corresponding sequence of stalks is

$$0 \to \mu_n(A) \to A^\times \xrightarrow{t \mapsto t^n} A^\times \to 0.$$

This sequence is clearly left-exact. To show it is exact, we must show that every element $a \in A^\times$ has an nth root. The derivative of the polynomial $p(t) = t^n - a \in A[t]$ is nt^{n-1}, which has nonzero image $\overline{p} \in (A/\mathfrak{m})[t]$ because $n \neq 0$ in $A/\mathfrak{m}$. Thus $\overline{p}$ is separable. Since A is strictly Henselian, p splits into linear factors. In particular, this implies that a possesses an nth root.

Example 2.6 (The Artin-Schreier sequence)**.** Let X be a scheme such that every residue field of X has characteristic $p \neq 0$. (For example, X could be an algebraic variety over a field k of characteristic p.) We define the **Artin-Schreier sequence** to be the sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \xrightarrow{t \mapsto t^p - t} \mathbb{G}_a \to 0.$$

Again, to prove exactness, we must check exactness of

$$0 \to \mathbb{Z}/p\mathbb{Z} \to A \xrightarrow{t \mapsto t^p - t} A \to 0$$

for every $A = \mathcal{O}_{X,\overline{x}}$. Again, this is clearly left-exact. We note that , for any $a \in A$, the polynomial $p(t) = t^p - t - a$ has derivative $pt^p - 1$, which maps to $-1 \neq 0$ in

$(A/\mathfrak{m})[t]$. Thus, by the reasoning above, a is in the image of $t \mapsto t^p - t$, and so the sequence is exact.

Note the geometric significance of these results. For example, the exactness of the Kummer sequence says that étale-locally, every invertible function a on some affine open set $U = \operatorname{Spec} B$ in X has an nth root. But this follows immediately from the fact that the map $B \to C = B[t]/(t^n - a)$ defines an étale cover; the function t on $\operatorname{Spec} C$ is an nth root of A.

References

1. James S. Milne, *Lectures on Etale Cohomology*, http://www.jmilne.org/math/, 2008.